INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2008; **56**:1489–1495
Published online 7 January 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1721

Two-temperature Euler equations for a plasma with slowing down of suprathermal particles

R. Sentis*,† and F. Chaland

CEA/Bruyeres, B.P. 12, 91680 Bruyeres, France

SUMMARY

In the hot plasma simulations when the main flow is modeled by two-temperature Euler equations, it is often useful to deal with slowing down of suprathermal particles created by fusion. Then a kinetic equation has to be addressed for these particles. We focus in this paper on the coupling between the fluid model and the kinetic equation; specially details for a coherent treatment of the electrostatic field are given. We emphasize some details of the numerical simulations and we give numerical results. Copyright © 2008 John Wiley & Sons, Ltd.

Received 26 April 2007; Revised 19 November 2007; Accepted 19 November 2007

KEY WORDS: coupling kinetic/fluid; two-temperature Euler equations; Monte Carlo method; Fokker–Planck operator; kinetic equation; Monte Carlo estimates

1. INTRODUCTION

In hot plasmas such as stellar plasmas or plasmas produced by laser in inertial confinement fusion, the fusion of deuterium and tritium creates helium ions whose initial velocity is very large compared with the thermal velocity of plasma ions (hence, they are called *suprathermal* (ST) *particles*). It is crucial to deal correctly with the slowing down of these particles due to the Coulomb interactions with the plasma and to perform the coupling of these phenomena with the hydrodynamics of the plasma. Moreover, it is well known that for the plasma flow simulation, one has to take into account a two-temperature model, one for the electrons and one for the ions. For a relevant physical modeling, one has to consider also the plasma electrostatic field $\mathbf{E}$ (see [1]).

The modeling of the transport of the ST particles by a Vlasov–Fokker–Planck equation and the numerical simulation have been studied for a very long time by physicists see, for example, [2, 3]

*Correspondence to: R. Sentis, CEA/Bruyeres, B.P. 12, 91680 Bruyeres, France.
†E-mail: remi.sentis@cea.fr

specially in a homogeneous plasma. However, the momentum and energy deposition in the momentum and energy balance equations of the fluid model has to be considered precisely and a coherent treatment of the plasma electrostatic field has to be made (which was not the purpose of the mentioned literature).

Section 1 is devoted to the statement of the model, especially the coupling aspects between the plasma and the ST particles. In Section 2, we give some enlightments on the Monte Carlo method for the ST particles. In the last section, some numerical results are given.

2. THE MODEL

Notations: Z is the ionization level of the plasma ions (the plasma is assumed to contain only one species of ions); m_0, m_s the mass of the ions and the ST particles; q_s the ST particle charge; N the plasma density; $\mathbf{U}$ the plasma velocity; ε_I and $\varepsilon_\mathrm{e}=\frac{3}{2}ZT_\mathrm{e}$ are the ion and electron internal energies; T_e the electron temperature; P_I and P_e the ion and electron pressures (the relation between P_* and ε_* is given by a perfect gas law); $P_\mathrm{tot}=P_\mathrm{e}+P_\mathrm{I}$. Denote also by $f(t,x,\mathbf{v})$ the distribution function of the ST particles, where $\mathbf{v}\in\mathbf{R}^3$ and x belongs to a bounded domain in $\mathbf{R}^3$.

The Vlasov–Fokker–Planck equation: For the sake of simplicity, one assumes that the ST particles undergo Coulomb interactions on electron population only; hence, the evolution equation of f reads

$$\frac{\partial}{\partial t}f+\mathbf{v}\cdot\nabla f=-\frac{q_\mathrm{s}}{m_\mathrm{s}}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}}+ZN\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{S}_\mathrm{E}f)\quad \text{with } (\mathbf{S}_\mathrm{E}f)(\mathbf{v})=(\mathbf{S}\widetilde{f})(\mathbf{v}-\mathbf{U}) \tag{1}$$

knowing that $\widetilde{f}(\mathbf{w})=f(\mathbf{w}+\mathbf{U})$ and that the simplest form of the operator $\mathbf{S}$ reads as

$$\mathbf{S}\widetilde{f}(\mathbf{w})=Y\mathbf{w}\widetilde{f}+O_\mathrm{e}(\mathbf{w})\cdot\frac{\partial\widetilde{f}}{\partial\mathbf{w}},\quad O_\mathrm{e}(\mathbf{w})\simeq Y\frac{T_\mathrm{e}}{m_\mathrm{s}}\frac{3}{2|\mathbf{w}|}\left(1-\frac{\mathbf{w}\mathbf{w}}{|\mathbf{w}|^2}\right)$$

The coefficient Y is roughly speaking proportional to $T_\mathrm{e}^{-3/2}$, see [3]. The change of variables $\mathbf{w}=\mathbf{v}-\mathbf{U}$ is performed throughout the paper. For any function ϕ defined on $\mathbf{R}^3$, one sets $\langle\phi\rangle=\int\phi(\cdot)\,\mathrm{d}\cdot$. Then balance relations related to (1) are

$$m_\mathrm{s}\left(\frac{\partial}{\partial t}\langle\mathbf{v}f\rangle+\nabla\cdot\langle\mathbf{v}\mathbf{v}f\rangle\right)=q_\mathrm{s}\mathbf{E}\langle f\rangle-m_\mathrm{s}ZN\langle\mathbf{S}\tilde{f}\rangle$$

$$\frac{m_\mathrm{s}}{2}\frac{\partial}{\partial t}\langle|\mathbf{v}|^2f\rangle+\frac{m_\mathrm{s}}{2}\nabla\cdot\langle\mathbf{v}|\mathbf{v}|^2f\rangle=q_\mathrm{s}\langle\mathbf{v}f\rangle\cdot\mathbf{E}-m_\mathrm{s}ZN\langle\mathbf{v}\cdot\mathbf{S}_\mathrm{E}f\rangle$$

The Euler system: Consider now the plasma model. The continuity equation is not changed by coupling with the ST particles

$$\frac{\partial}{\partial t}N+\nabla(N\mathbf{U})=0 \tag{2}$$

DOI: 10.1002/fld

In the momentum equation, one has to add by a natural way the counterpart of the momentum change of the ST particles ($m_s ZN\langle \mathbf{S}\tilde{f}\rangle$) added to ($q_s\mathbf{E}\langle f\rangle$), that is

$$m_0\left(\frac{\partial}{\partial t}+\nabla(\mathbf{U}\cdot)\right)(N\mathbf{U})+\nabla P_{\text{tot}}=m_s ZN\langle \mathbf{S}\tilde{f}\rangle-q_s\mathbf{E}\langle f\rangle \tag{3}$$

Since $\langle \mathbf{v}\cdot\mathbf{S}\tilde{f}(\cdot-\mathbf{U})\rangle=\langle(\mathbf{w}+\mathbf{U})\cdot\mathbf{S}\tilde{f}\rangle$, so with $\mathscr{W}=N(\varepsilon_e+\varepsilon_I+m_0|\mathbf{U}|^2/2)$, the plasma energy balance equation reads as

$$\left(\frac{\partial}{\partial t}+\nabla(\mathbf{U}\cdot)\right)\mathscr{W}+\nabla\cdot(P_{\text{tot}}\mathbf{U})+\mathscr{C}(T_e)=m_s ZN(\langle\mathbf{w}\cdot\mathbf{S}\tilde{f}\rangle+\mathbf{U}\cdot\langle\mathbf{S}\tilde{f}\rangle)-q_s\mathbf{E}\cdot\langle\mathbf{v}f\rangle$$

where $\mathscr{C}(T_e)$ is a diffusion operator related to the Spitzer thermal conduction, see [4]. Then, if only one internal energy would be addressed, the source term in the internal energy equation would reduce to $m_s ZN\langle\mathbf{w}\cdot\mathbf{S}\tilde{f}\rangle+Q$, where

$$Q=-q_s\mathbf{E}\cdot\langle\mathbf{w}\tilde{f}\rangle$$

As a matter of fact, two energy evolution equations are to be considered for a classical modeling of the plasma. Let us recall that without any coupling with the ST particles, they read as (see [5])

$$\left(\frac{\partial}{\partial t}+\nabla(\mathbf{U}\cdot)\right)(N\varepsilon_I)+P_I\nabla\cdot\mathbf{U}-\Omega=0 \tag{4}$$

$$\left(\frac{\partial}{\partial t}+\nabla(\mathbf{U}\cdot)\right)(N\varepsilon_e)+P_e\nabla\cdot\mathbf{U}+\mathscr{C}(T_e)+\Omega=0 \tag{5}$$

where Ω denotes to energy exchange term between ions and electrons (proportional to the difference of ion and electron temperatures). Consider now the coupling with the ST particles; the terms coming from this coupling has to be added to the right-hand side of (5)

$$\left(\frac{\partial}{\partial t}+\nabla(\mathbf{U}\cdot)\right)(N\varepsilon_e)+P_e\nabla\cdot\mathbf{U}+\mathscr{C}(T_e)+\Omega=m_s ZN\langle\mathbf{w}\cdot\mathbf{S}\tilde{f}\rangle+Q \tag{6}$$

To the best of our knowledge, system (2)–(6) has not been considered up to now (in [6] such a system is expressed in 1D without the term Q). The term Q is the counterpart of the work of the electric field; note that it has to be evaluated in the matter reference frame.

The coupling terms: The simplest definition for $\mathbf{E}$ is $\nabla P_e+ZNq_e\mathbf{E}=0$, see [1]. The momentum deposition within the matter reference frame reads

$$\langle\mathbf{S}\tilde{f}\rangle=Y\left(\langle\mathbf{w}\tilde{f}\rangle+3\frac{T_e}{m_s}\left\langle\frac{2\mathbf{w}}{|\mathbf{w}|^2}\tilde{f}\right\rangle\right)$$

The ST particle energy $m_s|\mathbf{w}|^2$ is generally large compared with the electron temperature; hence, the second term is negligible compared with the first one and $\langle\mathbf{S}\tilde{f}\rangle\simeq Y\langle\mathbf{w}\tilde{f}\rangle$. Since $\mathbf{w}\cdot O_e(\mathbf{w})=0$, the energy deposition term within the matter reference frame reads as $\langle\mathbf{w}\cdot\mathbf{S}\tilde{f}\rangle=Y\langle|\mathbf{w}|^2\tilde{f}\rangle$.

DOI: 10.1002/fld

3. NUMERICAL METHOD

The Euler system: One uses a Lagrange-type code based on the classical Wilkins method.

At each time step, this method consists of two stages: firstly one moves each node according to the force due to the pressure gradient, secondly one solves the internal energy equations, see [5]. These stages are followed by a mesh regularization.

To perform the coupling between the two models, one must evaluate in each cell the electric field $\mathbf{E}$ on one hand and the momentum and energy deposition by the ST particles on the other hand.

The transport equation: A Monte Carlo method is used. The method is based on the approximation of the solution $f(t,x,\mathbf{v})$ by a sum of Dirac measures, see [7]. For instance, at the beginning of the simulation, one sets

$$f(0,x,\mathbf{v})\,\mathrm{d}x\,\mathrm{d}\mathbf{v} \simeq \sum_{p=1}^{n_{\text{part}}} \omega_p(0)\delta_{\mathbf{v}_p}(\mathrm{d}\mathbf{v})\delta_{x_p}(\mathrm{d}x)$$

where n_{part} is the total number of particles. For each particle, $\omega_p(t)$ denotes its weight at time t; $x_p(t)$, $\mathbf{v}_p(t)$ its position and its velocity. At each time t, in each cell M, one has the following estimates:

$$\int_M \int f(t,x,\mathbf{v})\,\mathrm{d}\mathbf{v}\,\mathrm{d}x \simeq \sum_{p,\text{s.t. } x_p\in M} \omega_p, \quad \int_M \int f(t,x,\mathbf{v})\mathbf{v}\,\mathrm{d}\mathbf{v}\,\mathrm{d}x \simeq \sum_{p,\text{s.t. } x_p\in M} \omega_p \mathbf{v}_p$$

It is useful for a good implementation to have a probabilistic interpretation of the dual operator of

$$f \hookrightarrow -\frac{q_{\text{s}}}{m_{\text{s}}}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}} + ZN\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{S}_E f)$$

that is to say (with $Y_D = \frac{3}{2}ZNYT_{\text{e}}/m_{\text{s}}$)

$$\varphi \hookrightarrow \frac{q_{\text{s}}}{m_{\text{s}}}\mathbf{E}\cdot\frac{\partial \varphi}{\partial \mathbf{v}} - ZNY\mathbf{w}\frac{\partial \varphi}{\partial \mathbf{v}} + Y_{\text{D}}\frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{|\mathbf{w}|}\left(1-\frac{\mathbf{w}\mathbf{w}}{|\mathbf{w}|^2}\right)\frac{\partial \varphi}{\partial \mathbf{v}}\right)$$

Note that

1. $\mathbf{E}(\partial/\partial\mathbf{v})\varphi$ corresponds to an acceleration in the direction of $\mathbf{E}$;
2. $-\mathbf{w}(\partial/\partial\mathbf{v})\varphi$ corresponds to a straight line slowing down (in the matter reference frame);
3. the deflection operator $(\partial/\partial\mathbf{w})(1/|\mathbf{w}|)((1-\mathbf{w}\mathbf{w}/|\mathbf{w}|^2)\partial\varphi/\partial\mathbf{w})$ corresponds to a diffusion on a sphere, indeed one can check that the solution of the elementary equation (of Laplace–Beltrami type)

$$\frac{\partial \varphi}{\partial t} - \frac{\partial}{\partial \mathbf{w}}\left(\frac{1}{|\mathbf{w}|}\left(1-\frac{\mathbf{w}\mathbf{w}}{|\mathbf{w}|^2}\right)\frac{\partial \varphi}{\partial \mathbf{w}}\right) = 0, \quad \varphi(0,\mathbf{w}) = \delta_{\mathbf{w}_0}$$

satisfies $\int \varphi(t,\mathbf{w})|\mathbf{w}|^2\,\mathrm{d}\mathbf{w} = |\mathbf{w}_0|^2$ for any t. As a matter of fact, the solution of this equation is an analytic function depending only on the angular variable $\mathbf{w}\cdot\mathbf{w}_0$ and its support is the sphere of radius $|\mathbf{w}_0|$.

DOI: 10.1002/fld

Hence, the Monte Carlo method consists in a tracking of the particles in the mesh used by the hydrodynamics solver. In each cell M, where the mean velocity is $\mathbf{U}_M$, the particles move with their relative velocities $\mathbf{w}_p = \mathbf{v}_p - \mathbf{U}_M$, and their velocities are changed according to the three modifications listed above.

Moreover, when the particle p goes from cell M to cell M', its velocity $\mathbf{w}_p$ has to be corrected by the following way:

$$\mathbf{w}'_p + \mathbf{U}_{M'} = \mathbf{w}_p + \mathbf{U}_M$$

In each cell M, one has to estimate the quantities $\langle \tilde{f} \rangle|_M, \langle \mathbf{w} \tilde{f} \rangle|_M, \langle |\mathbf{w}|^2 \tilde{f} \rangle|_M$, in the matter reference frame. For instance, we obtain $\langle \mathbf{w} \tilde{f} \rangle|_M \simeq \sum_{p,\text{s.t. } x_p \in M} \mathbf{w}_p L_p^M \omega_p$, where L_p^M denotes the distance covered by the particle p in the cell M.

It is well known that the accuracy of the Monte Carlo method is proportional to $1/\sqrt{n_{\text{part}}}$; hence, variance reduction techniques are to be used. The basic technique is the so-called *Russian Roulette*, which consists in killing the particles when their weights are small enough (with respect to their initial values). Of course, it is necessary to create particles with an agreement to the physical source localization. Other techniques like *zone spitting* may be used, see [7], for instance.

4. NUMERICAL RESULTS

Numerical example 1

One addresses a dense and hot spherical plasma with a source of ST particles in the center of the sphere. The initial density and temperature profiles are very stiff (conditions of inertial confinement fusion plasma), see Figure 1. At a given time, one compares the profile of the momentum deposition with the profile of ∇P_{tot}, (Figure 2). One notes that the momentum deposition $m_s N \langle \mathbf{S} \tilde{f} \rangle$ is important in the zone where the pressure gradient is large, which is the crucial zone for energy deposition.

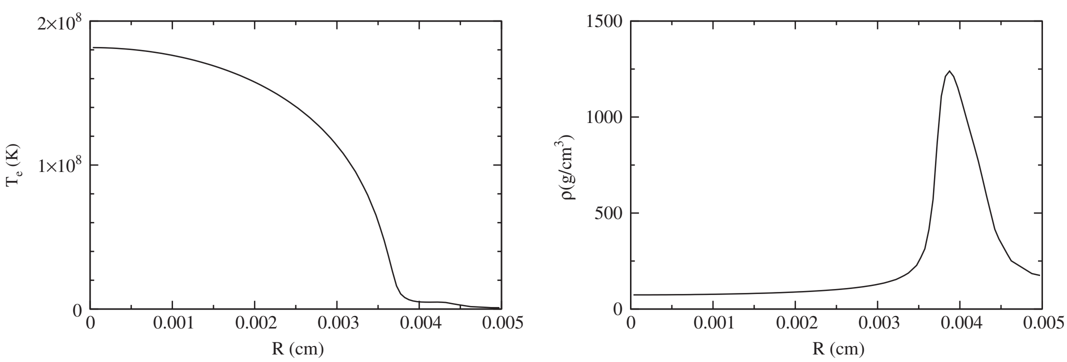

Figure 1. Temperature and density profiles *versus* radius.

DOI: 10.1002/fld

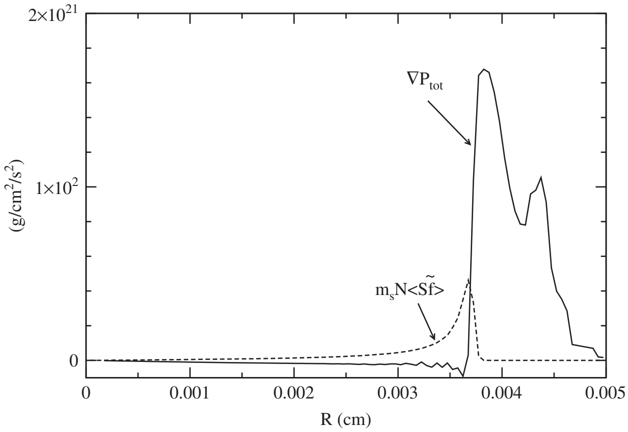

Figure 2. Profiles of ∇P_{tot} and of $m_{\mathrm{S}} N \langle \mathbf{S} \tilde{f} \rangle$ *versus* radius.

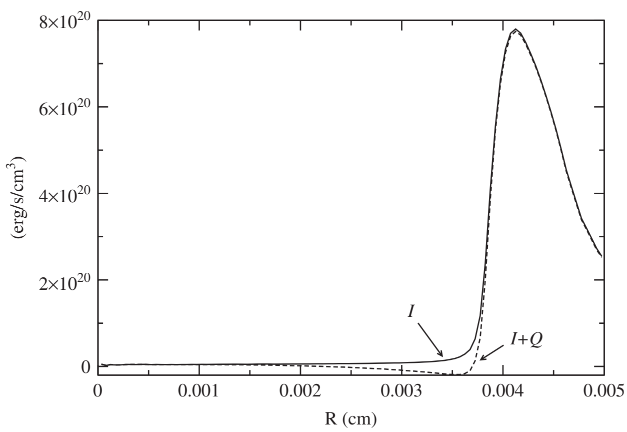

Figure 3. Profiles of I and $I+Q$ *versus* radius.

Numerical example 2

The same spherical plasma is considered with the same T_{e} profile, but with N much lower. The comparison of Q with $I = m_{\mathrm{S}} N \langle \mathbf{w} \cdot \mathbf{S} \tilde{f} \rangle$ at a given time step is plotted in Figure 3. One sees that in that case the influence of the term Q may be not negligible.

5. CONCLUSION

In the two-temperature Euler equations modeling a hot plasma, we have performed the coupling with a simplified transport equation which is relevant for the slowing down of the ST particles.

DOI: 10.1002/fld

This coupling has been made in a consistent manner in such a way that there is good momentum and energy balance. It is implemented in a 2D plasma code and some preliminary numerical results are given.

REFERENCES

1. Chen FF. *Introduction to Plasma Physics*. Plenum Press: New York, 1977.
2. Landau L. *The Transport Equation in the Case of Coulomb Interactions*. Pergamon Press: Oxford, 1965 (in collected paper of L. Landau).
3. Shkarofsky IP, Johnston TW, Bachynski MP. *The Particle Kinetics of Plasma*. Addison-Wesley: Reading, MA, 1966.
4. Spitzer L. *Physics of Fully Ionized Gases*. Interscience: New York, 1962.
5. Bowers RL, Wilson JR. *Numerical Modeling in Applied Physics*. Jones-and-Barlett: Boston, MA, 1991.
6. Kuroki Y *et al.* Fusion product momentum deposition. *Nuclear Fusion* 2000; **40**:357.
7. Lapeyre B, Pardoux E, Sentis R. *Introduction aux Méthodes de Monte-Carlo Pour les Equations de Transport*. Springer: Berlin, 1997 or *Monte-Carlo Methods for Transport and Diffusion*. Oxford University Press: Oxford, 2003.

DOI: 10.1002/fld